Exact results for the infinite supersymmetric extensions of the infinite square well

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Abstract

One-dimensional potentials defined by $V^{(S)}(x) = S(S+1)\hbar^2\pi^2/[2ma^2\sin^2(\pi x/a)]$ (for integer S) arise in the repeated supersymmetrization of the infinite square well, here defined over the region $(0,a)$. We review the derivation of this hierarchy of potentials and then use the methods of supersymmetric quantum mechanics, as well as more familiar textbook techniques, to derive for the first time new compact closed-form expressions for the normalized solutions, $\psi_n^{(S)}(x)$, for all $V^{(S)}(x)$ in terms of well-known special functions. We then suggest additional avenues for research questions related to, and pedagogical applications of, these solutions, including the behavior of the corresponding $\phi_n^{(S)}(p)$ for large $|p|$ and general questions about the supersymmetric hierarchies of potentials which include an infinite barrier.

I. INTRODUCTION

Generations of physicists have been trained in quantum mechanics by repeated practice using a handful (literally, five or so) of familiar model systems, including the infinite square well (hereafter ISW), harmonic oscillator (HO), hydrogen atom, and rigor rotator. Some of these systems, including many with a high degree of symmetry, are amenable to operator methods, such as the raising and lowering operator approach for the HO. These systems, however, are typically solved using standard techniques involving the solution of the Schrödinger equation in position space, enforcing the appropriate boundary conditions to give the quantized energy levels, and often finding solutions in terms of special functions. Given the very limited number of such tractable examples, it can be a welcome addition to the literature of introductory quantum mechanics to find an entirely new class of potentials which can be approached (and solved completely) using a variety of such methods.

One method of obtaining 'new potentials from old' is the use of supersymmetric quantum mechanics[1–8] (SUSYQM) which can generate new model systems which have the same energy eigenvalue spectrum, save for missing the ground-state energy in the new 'supersymmetrized' version of the potential, the so-called partner or superpartner potential. For the hydrogen atom, the supersymmetrization leads to a hierarchy[4,9,10] of potentials all related to each other by the supersymmetrization process.

It has been known for some time (if not generally appreciated) that the most familiar of all model systems, the infinite square well, has just such a hierarchy[4,9] of superpartner potentials (with energy spectra related by supersymmetry) with a very simple form, namely

$$V^{(S)}(x) = \frac{S(S+1)\hbar^2\pi^2}{2ma^2\sin^2(\pi x/a)} \qquad \text{for } 0 < x < a\,, \tag{1}$$

where the original potential $V^{ISW}(x)$ (or $S \equiv 0$ case) is defined over the interval $(0, a)$. In this notation $S = 1$ corresponds to the first supersymmetrization of the ISW, $S = 2$ the result of supersymmetrizing $V^{(S=1)}(x)$, and so forth. Beyond a mention of the ground-state wave functions for this class of potentials,[4,10] we have found no detailed discussions of the solutions for this hierarchy, in the research, mathematical physics, or pedagogical literature. For that reason, an exploration of this system, both in the derivation of the solutions, and their interpretation, is the topic of this work where we present (for the first time) closed-form expressions for all solutions, $\psi_n^{(S)}(x)$, of this problem in terms of known special functions. We emphasize throughout the interplay of the application of formal methods from supersymmetric quantum mechanics, more standard (textbook level) approaches using differential equations, and the use of symbolic manipulation tools as pedagogical methods in approaching this problem.

In the next section (II) we review the methods of supersymmetric quantum mechanics and then in Sec. III we revisit the derivation of the fact that the potentials in Eq. (1) are the result of repeated supersymmetrization of the ISW, but also showing how exact normalized solutions to the general S case can be obtained by iteration. In Sec. IV we use standard textbook methods based on differential equations to derive compact, closed-form solutions for this hierarchy of potentials, for general S, which are written in terms of the Gegenbauer polynomials. For $S = 0$, this expression also results in a new formulation of the usual ISW wave functions

$$\psi_n^{(S=0)}(x) = \psi_n^{\rm ISW}(x) = \sqrt{\frac{2}{a}}\,\mathcal{U}_{n-1}[\cos(\pi x/a)]\ \sin(\pi x/a)\ , \tag{2}$$

written in terms of $\mathcal{U}_n[z]$, the Chebyshev polynomials of the second kind. Finally, in Sec. V, we briefly discuss avenues for further exploration of this rich system, as well as open questions regarding the hierarchies of supersymmetric extensions of other familiar one-dimensional systems containing infinite wall potentials.

II. SUPERSYMMETRIC QUANTUM MECHANICS

Factorization methods[11,12] have historically proved to be powerful tools in the solution of a wide variety of problems in quantum mechanics and mathematical physics, especially those with a high degree of symmetry. The connection to supersymmetry[1–8,13] (hereafter SUSY) and the interest in iso-spectral Hamiltonian systems has provided further motivation for using such approaches in a variety of one-dimensional model systems.

We begin by assuming a generic one-dimensional potential, $V(x)$, admitting a non-degenerate ground-state solution, $\psi_0(x)$, with energy, E_0. If we define a shifted potential energy function, $V^{(-)}(x) \equiv V(x) - E_0$, we know, by construction, that $\psi_0(x)$ satisfies

$$\hat{H}^{(-)}\psi_0(x) \equiv \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V^{(-)}(x)\right]\psi_0(x) = 0\ , \tag{3}$$

and that $V^{(-)}(x)$ has a zero-energy ($E_0^{(-)} = 0$) ground-state. Since $\psi_0(x)$ is assumed known, we can use Eq. (3) to then write $\hat{H}^{(-)}$ in the form

$$\hat{H}^{(-)} = \frac{\hbar^2}{2m}\left[-\frac{d^2}{dx^2} + \frac{\psi_0''(x)}{\psi_0(x)}\right] . \tag{4}$$

If we define the ladder operators

$$\hat{A} \equiv \frac{\hbar}{\sqrt{2m}}\left(\frac{d}{dx} - \frac{\psi_0'(x)}{\psi_0(x)}\right) \qquad \text{so that} \qquad \hat{A}^\dagger \equiv \frac{\hbar}{\sqrt{2m}}\left(-\frac{d}{dx} - \frac{\psi_0'(x)}{\psi_0(x)}\right) , \tag{5}$$

we then have $\hat{A}^\dagger \hat{A} = \hat{H}^{(-)}$ and $\hat{H}^{(-)}$ is factorizable.

While $\hat{A}^\dagger \hat{A} = \hat{H}^{(-)}$ now factorizes the original Hamiltonian (up to an additive constant, $-E_0$), the related combination, $\hat{A}\hat{A}^\dagger$, can be seen to define an (in principle) entirely new potential, since

$$\hat{A}\hat{A}^\dagger \equiv \hat{H}^{(+)} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V^{(+)}(x) , \tag{6}$$

where

$$V^{(+)}(x) = V^{(-)}(x) - \frac{\hbar^2}{m}\frac{d}{dx}\left[\frac{\psi_0'(x)}{\psi_0(x)}\right] = -V^{(-)}(x) + \frac{\hbar^2}{m}\left[\frac{\psi_0'(x)}{\psi_0(x)}\right]^2 . \tag{7}$$

If $\psi_n^{(-)}(x)$ is any eigenfunction of $\hat{H}^{(-)}$ with eigenvalue $E_n^{(-)}$, then $\hat{A}\psi_n^{(-)}(x)$ is an eigenfunction of $\hat{H}^{(+)}$ with the same eigenvalue. This is easily seen since

$$\hat{H}^{(+)}\left(\hat{A}\psi_n^{(-)}\right) = \hat{A}\hat{A}^\dagger\left(\hat{A}\psi_n^{(-)}\right) = \hat{A}\left(\hat{H}^{(-)}\psi_n^{(-)}\right) = E_n^{(-)}\left(\hat{A}\psi_n^{(-)}\right) .$$

Similarly one can show that if $\psi_n^{(+)}(x)$ is an eigenfunction of $\hat{H}^{(+)}$ with eigenvalue $E_n^{(+)}$, then $\hat{A}^\dagger\psi_n^{(+)}(x)$ is an eigenfunction of $\hat{H}^{(-)}$ with the same eigenvalue. Taken together, these relations can be shown to imply[13]

$$E_n^{(+)} = E_{n+1}^{(-)} , \quad \psi_n^{(+)}(x) = \frac{1}{\sqrt{E_{n+1}^{(-)}}}\hat{A}\,\psi_{n+1}^{(-)}(x) , \quad \text{and} \quad \psi_{n+1}^{(-)}(x) = \frac{1}{\sqrt{E_n^{(+)}}}\hat{A}^\dagger\,\psi_n^{(+)}(x) . \tag{8}$$

Thus, the two systems defined by $V^{(\pm)}(x)$ have the same energy spectrum, $E_n^{(\pm)}$, except that the zero-energy ground-state of $V^{(-)}(x)$ has no counterpart in $V^{(+)}(x)$. We also note that if the original $\psi_n^{(-)}(x)$ are orthogonal and normalized, then so are the $\psi_n^{(+)}(x)$, since using Eq. (8) we have

$$\begin{aligned}
\langle\psi_n^{(+)}|\psi_m^{(+)}\rangle &= \frac{1}{\sqrt{E_{n+1}^{(-)}E_{m+1}^{(-)}}}\langle\psi_{n+1}^{(-)}|\hat{A}^\dagger\,\hat{A}|\psi_{m+1}^{(-)}\rangle \\
&= \frac{1}{\sqrt{E_{n+1}^{(-)}E_{m+1}^{(-)}}}\langle\psi_{n+1}^{(-)}|\hat{H}^{(-)}|\psi_{m+1}^{(-)}\rangle \\
&= \sqrt{\frac{E_{m+1}^{(-)}}{E_{n+1}^{(-)}}}\,\langle\psi_{n+1}^{(-)}|\psi_{m+1}^{(-)}\rangle = \delta_{n,m} .
\end{aligned} \tag{9}$$

As an example, we note that the simplest SUSYQM version of a familiar one-dimensional system is the harmonic oscillator (HO), with potential energy and energy eigenvalues given by

$$V^{\rm HO}(x) = \frac{1}{2}m\omega^2 x^2 \qquad \text{and} \qquad E_n = (n+1/2)\hbar\omega\,, \tag{10}$$

where $n = 0, 1, 2....$ The energy-eigenstate solutions are well-known to be

$$\psi_n(x) = C_n H_n(v)\, e^{-v^2/2}\,, \tag{11}$$

where $v \equiv x/\beta$, $\beta \equiv \sqrt{\hbar/m\omega}$, the $H_n(v)$ are the Hermite polynomials, and the C_n are normalization constants given by $C_n = 1/\sqrt{\beta\sqrt{\pi}2^n n!}$.

To apply the methods above, we first 'zero out' the potential and energy eigenvalues by subtracting $E_0 = \hbar\omega/2$ from both to obtain

$$V^{(-)}(x) = \frac{1}{2}m\omega^2 x^2 - \frac{\hbar\omega}{2} \qquad \text{and} \qquad E_n^{(-)} = n\hbar\omega\,. \tag{12}$$

Then since $\psi_0(x) \propto e^{-x^2/2\beta^2}$ we find that the superpartner potential is

$$V^{(+)}(x) = -V^{(-)}(x) + \frac{\hbar^2}{m}\left(-\frac{x}{\beta^2}\right)^2 = \left(-\frac{1}{2}m\omega^2 x^2 + \frac{\hbar\omega}{2}\right) + m\omega^2 x^2 = \frac{1}{2}m\omega^2 x^2 + \frac{\hbar\omega}{2}\,, \tag{13}$$

with corresponding energies given by

$$E_n^{(+)} = E_{n+1}^{(-)} = (n+1)\hbar\omega = (n+1/2)\hbar\omega + \frac{\hbar\omega}{2}\,. \tag{14}$$

We see that up to the common constant energy term, $\hbar\omega/2$, $V^{(+)}(x)$ and $E_n^{(+)}$ are exactly the same as for the original harmonic oscillator system, so that the supersymmetric partner potentials are in fact identical. Using the results of Eq. (8) for the wave functions, we find that

$$\psi_n^{(+)}(x) = \frac{1}{\sqrt{E_{n+1}^{(-)}}}\hat{A}\,\psi_{n+1}^{(-)}(x) = \frac{1}{\sqrt{n+1}}\left(\frac{\beta}{\sqrt{2}}\right)\left(\frac{d}{dx} + \frac{x}{\beta^2}\right)\psi_{n+1}^{(-)}(x)\,, \tag{15}$$

which is equivalent to the standard textbook lowering operator relation,

$$\sqrt{n+1}\,|n\rangle = \hat{A}|n+1\rangle \qquad \text{where} \qquad \hat{A} \equiv \frac{1}{\sqrt{2m\hbar\omega}}[i\hat{p} + m\omega x]\,. \tag{16}$$

III. SUPERSYMMETRIC VERSIONS OF THE INFINITE SQUARE WELL

In contrast to the harmonic oscillator, the supersymmetric partner of the infinite square well (ISW), is non-trivially different, so we begin by examining the first 'SUSY extension' of the ISW, which we label by $S = 1$. We define the ISW potential by

$$V^{\rm ISW}(x) \equiv \begin{cases} \infty & \text{for } x < 0 \text{ or } x > a \\ 0 & \text{for } 0 < x < a \end{cases} \tag{17}$$

and label the energy eigenstates and eigenvalues as

$$\psi_n^{\rm ISW}(x) \equiv \psi_n(x) = \sqrt{\frac{2}{a}} \sin\left[\frac{(n+1)\pi x}{a}\right] \qquad \text{and} \qquad E_n = \frac{\hbar^2\pi^2}{2ma^2}(n+1)^2 \,, \tag{18}$$

where $n = 0, 1, 2, ...$ so that the label $n = 0$ corresponds to the ground-state, to be consistent with the notation in Sec. II. A parameter that will appear often in subsequent expressions is the zero-point energy of the ISW, which we will define as $\mathcal{E}_0 \equiv E_0 = \hbar^2\pi^2/2ma^2$. By subtracting this zero-point energy from the potential and energy eigenvalues, we have

$$V^{(-)}(x) = V^{ISW}(x) - \mathcal{E}_0 \qquad \text{and} \qquad E_n^{(-)} = \mathcal{E}_0\left[(n+1)^2 - 1\right] . \tag{19}$$

The ground-state wave function is $\psi_0(x) = \sqrt{2/a}\sin(\pi x/a)$ which gives

$$\left(\frac{\psi_0'(x)}{\psi_0(x)}\right)^2 = \frac{\pi^2}{a^2}\frac{\cos^2(\pi x/a)}{\sin^2(\pi x/a)} \,, \tag{20}$$

so that using Eq. (7) and Eq. (20), the partner potential to $V^{(-)}(x)$ is

$$V^{(+)}(x) = \mathcal{E}_0 \frac{2}{\sin^2(\pi x/a)} - \mathcal{E}_0 \,, \tag{21}$$

with corresponding energies given by

$$E_n^{(+)} = E_{n+1}^{(-)} = \mathcal{E}_0(n+2)^2 - \mathcal{E}_0 \,. \tag{22}$$

Just as in the harmonic oscillator case, since both $V^{(-)}(x)$ and $E_n^{(-)}$ have the common factor of $-\mathcal{E}_0$, we can rescale the zero of potential and quantized energies to find that the 'first supersymmetrization' (or $S = 1$ version) of the ISW can be re-defined as

$$V^{(S=1)}(x) = \frac{2\mathcal{E}_0}{\sin^2(\pi x/a)} = \frac{1(1+1)\mathcal{E}_0}{\sin^2(\pi x/a)} \quad \text{and} \quad E_n^{(S=1)} = \mathcal{E}_0(n+2)^2 \,. \tag{23}$$

This result has appeared in numerous journal articles,[6,13] monographs[2,14] and even has made its way into textbook problem sets.[15,16] Completely independently of the SUSYQM connection to the ISW, discussions of similar potentials have appeared in collections of quantum mechanics problems[17] and ultimately can trace its origin back to the (trigonometric) Pöschl-Teller potential[18] and at least one group[19] has explored the SUSY partners of that case.

Using the results of Eq. (8) we find the wave functions of the $S=1$ system to be

$$\begin{aligned}
\psi_n^{(S=1)}(x) &= \frac{1}{\sqrt{E_{n+1}^{(-)}}}\,\hat{A}\,\psi_{n+1}^{\rm ISW}(x) \\
&= \frac{a}{\pi}\frac{1}{\sqrt{(n+2)^2-1}}\left(\frac{d}{dx}-\frac{\pi}{a}\frac{\cos(\pi x/a)}{\sin(\pi x/a)}\right)\psi_{n+1}^{\rm ISW}(x) \\
&= \sqrt{\frac{2}{a}}\frac{1}{\sqrt{(n+2)^2-1}}\left\{(n+2)\cos\left[\frac{(n+2)\pi x}{a}\right]-\frac{\cos(\pi x/a)}{\sin(\pi x/a)}\sin\left[\frac{(n+2)\pi x}{a}\right]\right\}.
\end{aligned} \tag{24}$$

For comparison to results from the ISW (or $S=0$) case and higher S solutions, we note that for $S=1$ the $n=0,1$ solutions are (up to an arbitrary sign factor) given by

$$\psi_0^{(S=1)}(x) = 2\sqrt{\frac{2}{3a}}\sin^2\left(\frac{\pi x}{a}\right) \qquad \text{and} \qquad \psi_1^{(S=1)}(x) = \frac{4}{\sqrt{a}}\cos\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{\pi x}{a}\right). \tag{25}$$

One can now repeat the supersymmetrization procedure by acting on the $S=1$ solutions (using $\psi_0^{(S=1)}(x)$ from Eq. (25) as the new ground-state wave function) to obtain the supersymmetric partner potential (and their energy eigenvalues) corresponding to $S=2$. After again taking into account identical energy factors (common to both the potential and energies), we find that

$$V^{(S=2)}(x) = \mathcal{E}_0\frac{6}{\sin^2(\pi x/a)} = \mathcal{E}_0\frac{2(2+1)}{\sin^2(\pi x/a)} \quad \text{and} \quad E_n^{(S=2)} = \mathcal{E}_0(n+3)^2, \tag{26}$$

with the wave functions given by

$$\psi_n^{(S=2)}(x) = \frac{a}{\pi}\frac{1}{\sqrt{(n+3)^2-4}}\left(\frac{d}{dx}-\frac{2\pi}{a}\frac{\cos(\pi x/a)}{\sin(\pi x/a)}\right)\psi_{n+1}^{(S=1)}(x). \tag{27}$$

We see that this gives $\psi_0^{(S=2)}(x) \propto \sin^3(\pi x/a)$ and $\psi_1^{(S=2)}(x) \propto \cos(\pi x/a)\,\sin^3(\pi x/a)$ which can be compared to the results in Eq. (25).

One quickly recognizes the pattern, and by repeatedly applying the supersymmetrization procedures, one can show that the family of potentials generated in this hierarchy of supersymmetric extensions of the infinite square well (hereafter SISW) is given by[4,9]

$$V^{(S)}(x) = \mathcal{E}_0\frac{S(S+1)}{\sin^2(\pi x/a)} \qquad \text{and} \qquad E_n^{(S)} = \mathcal{E}_0(n+S+1)^2. \tag{28}$$

We illustrate this hierarchy of potential energy functions, with the corresponding energy spectra, in Fig. 1. We note that for quantized eneriges given by $k^2\mathcal{E}_0$ there are k different $V^{(S)}(x)$ potentials which will have that value as a possible state. All of the energy eigenvalues represented by dashed lines in Fig. 1 which are above the minimum value of a given $V^{(S)}(x)$ (namely $V^{(S)}_{\rm min} = V^{(S)}(x = a/2) = S(S+1)\mathcal{E}_0$) correspond to allowed states of that system.

The operator connection which generalizes the results from Eqns. (24) and (27) to connect the S and $S+1$ states is then

$$\begin{aligned}
\psi_n^{(S+1)}(x) &= \left(\frac{a}{\pi}\right) \frac{1}{\sqrt{(n+S+2)^2 - (S+1)^2}} \left[\frac{d}{dx} - \frac{(S+1)\pi}{a}\frac{\cos(\pi x/a)}{\sin(\pi x/a)}\right] \psi_{n+1}^{(S)}(x) \\
&\equiv \hat{B}^{(S)}\, \psi_{n+1}^{(S)}(x)\,, \qquad (29)
\end{aligned}$$

which defines the general operator $\hat{B}^{(S)}$, which is analogous to the SUSYQM operator $\hat{A}$, but made dimensionless.

This approach can then (in principle) be used to obtain the energy eigenstates of any (n, S) combination by iteratively using Eq. (29) as often as necessary to generate the desired state. For example, if one wants the $(n = 5, S = 7)$ state, one can repeatedly act on the $(n = 12, S = 0)$ state (i.e., the $n = 12$ ISW wave function) using the appropriate $\hat{B}^S$ operators, or more generally

$$\psi_n^{(S)}(x) = \prod_{p=1}^{p=S} \hat{B}^{(p)} \left[\psi_{n+S}^{\rm ISW}(x)\right] . \qquad (30)$$

One can, of course, implement this algorithm in symbolic manipulation programs to extract any desired solution very efficiently.

Because the supersymmetrization procedure respects the normalization of the wave functions, as shown in Eq. (9), we know that the $\psi_n^{(S)}(x)$ solutions will be appropriately normalized since the original $\psi_n^{(S=0)}(x) = \psi_n^{\rm ISW}(x)$ were. Using results obtained from this approach, we illustrate the lowest-lying quantum states ($n = 0, 1, 2$) for the first three values of S (including the $S = 0$ or ISW case) in Fig. 2 which exhibit the expected nodal structure. For larger S values, the probability density for low-n states is preferentially located the center of the well, and away from the walls at $x = 0, a$, in contrast to the more 'flat' distribution for the ISW. To visualize this limiting case, we plot some of the $\psi_n^{(S)}(x)$ in Fig. 3, where for fixed $n = 5$ we show $|\psi_n^{(S)}(x)|^2$ for two values of S ($S = 0, 10$). This more clearly illustrates the peaking of the quantum probability density near the classical turning points, namely where $E_n^{(S)} = V^{(S)}(x)$, in the non-trivial (non-ISW) $S > 0$ cases.

The result for the hierarchy of supersymmetric ISW (SISW) potentials in Eq. (28) was evidently first noted by Sukarmar,[9] and has also been discussed by others[4,10] who, in addition, showed that the ground-state wave functions (in our notation) are proportional to

$\psi_0^{(S)}(x) \propto \sin^{(S+1)}(\pi x/a)$. The ground-state wave functions are automatically generated by the repeated supersymmetrizations above and confirm this result, and we then easily find the completely normalized results for $\psi_0^{(S)}(x)$. One can also easily see that the first-excited states are proportional to $\cos(\pi x/a)\,\sin^{(S+1)}(\pi x/a)$ (again consistent with earlier results) and obtain the corresponding normalizations for them as well. In this way we find the universal result for the ground-state and first-excited state for the general S case is

$$\psi_0^{(S)}(x) \;=\; \frac{1}{\sqrt{a}}\left[\frac{\sqrt{\pi}\,\Gamma(S+2)}{\Gamma(S+3/2)}\right]^{1/2}\sin^{S+1}(y) \tag{31}$$

$$\psi_1^{(S)}(x) \;=\; \frac{1}{\sqrt{a}}\left[\frac{2\sqrt{\pi}\,\Gamma(S+3)}{\Gamma(S+3/2)}\right]^{1/2}\cos(y)\,\sin^{S+1}(y)\,, \tag{32}$$

where we will henceforward write $y \equiv \pi x/a$ for notational simplicity. These results can be confirmed by direct substitution into the Schrödinger equation for $V^{(S)}(x)$ and $E_n^{(S)}$ from Eq. (28), providing an example of the pedagogical use of many aspects of this rich problem. It is also easy to show that the $\psi_0^{(S+1)}(x)$ and $\psi_1^{(S)}(x)$ satisfy the operator relation $\psi_0^{(S+1)}(x) = \hat{B}^{(S)}\,\psi_1^{(S)}(x)$ in Eq. (29) for general S.

We note that for $x \approx 0$ (i.e. near the infinite wall), the potential for the general S case reduces to

$$V_{(S)}(x) \sim \frac{S(S+1)\hbar^2}{2mx^2}\,, \tag{33}$$

which is clearly similar in form to the standard 'centrifugal barrier' term arising from angular momentum considerations in 3D problems involving central potentials, namely $V_C(r) = l(l+1)\hbar^2/2mr^2$, here with the parameter S playing the role of the angular momentum quantum number l: a similar barrier term also arises near the other infinite wall at $x = a$.

To understand this behavior, we observe that for such an initial potential the wave function near a wall (say at $x = 0$) must have the form $\psi_0^{(S=0)}(x) = a_1 x + \mathcal{O}(x^2)$ so that the $S = 1$ potential will necessarily contain a term of the form

$$V^{(S=1)}(x\sim 0) = \frac{\hbar^2}{m}\left(\frac{\psi_0'(x)}{\psi_0(x)}\right)^2 \sim \frac{\hbar^2}{m}\left(\frac{a_1}{a_1 x}\right)^2 \propto \frac{2\hbar^2}{2mx^2}\,, \tag{34}$$

giving an $S(S+1) = 1(1+1) = 2$ 'centrifugal barrier' term in the first supersymmetric potential near $x = 0$. The $S = 1$ wave functions must then satisfy the appropriate boundary conditions at $x \sim 0$ (just as would 3D radial wave functions in central potentials), namely $\psi_0^{(S=1)}(x) = a_2 x^2 + \mathcal{O}(x^3)$ and using this dependence when one performs the second supersymmetrization, one finds

$$V^{(S=2)}(x) = \frac{6\hbar^2}{2mx^2}\,, \tag{35}$$

consistent with $S(S+1) = 2(2+1) = 6$. Once again, one can proceed by induction to derive the form in Eq. (33) and the fact that $\psi_n^{(S)}(x\sim 0) \propto x^{S+1}$. This behavior is clearly illustrated in Fig. 2 (lower-right frame) where $\psi_0^{(S)}(x)$ approaches the boundaries at $x = 0$ increasingly smoothly as S increases, as the wave function 'tunnels' into the 'angular-momentum-like' barriers near the walls.

IV. SISW WAVE FUNCTIONS

To explore the structure of the solutions in the combined (n, S) space, we first use the iterative procedures outlined above to collect the 5 lowest lying $S = 1$ solutions. Motivated by the forms in Eq. (25), we use symbolic manipulation software to expand and factor the resulting trigonometric functions in specific ways to obtain the following

$$\psi_0^{(S=1)}(x) = 2\sqrt{\frac{2}{3a}}\sin^2(y) \tag{36}$$

$$\psi_1^{(S=1)}(x) = \frac{4}{\sqrt{a}}\,[\cos(y)]\sin^2(y) \tag{37}$$

$$\psi_2^{(S=1)}(x) = 4\sqrt{\frac{2}{15a}}\left[-1 + 6\cos^2(y)\right]\sin^2(y) \tag{38}$$

$$\psi_3^{(S=1)}(x) = \frac{4}{\sqrt{3a}}\left[-3\cos(y) + 8\cos^3(y)\right]\sin^2(y) \tag{39}$$

$$\psi_4^{(S=1)}(x) = 2\sqrt{\frac{2}{35a}}\left[3 - 48\cos^2(y) + 80\cos^4(y)\right]\sin^2(y)\,, \tag{40}$$

where we again use the notation $y \equiv \pi x/a$. We have done this for higher values of S and find quite generally that all of the solutions for a given value of S have a common factor of $\sin^{(S+1)}(y)$ and that the remaining part of the wave function is a polynomial in $\cos(y)$ of order n. This uniform pattern for the $S \geq 1$ states seems, at first, to be rather different than the standard ISW results in Eq. (18) which corresponds to $S = 0$, at least until we realize that repeated use of trigonometric identities can be applied to the $\psi_n^{\rm ISW}(x) = \psi_n^{(S=0)}(x)$ to obtain the expressions

$$\begin{aligned}
\psi_0^{(S=0)}(x) &= \sqrt{\frac{2}{a}}\sin(y) && (41)\\
\psi_1^{(S=0)}(x) &= 2\sqrt{\frac{2}{a}}\,[\cos(y)]\sin(y) && (42)\\
\psi_2^{(S=0)}(x) &= \sqrt{\frac{2}{a}}\left[-1+4\cos^2(y)\right]\sin(y) && (43)\\
\psi_3^{(S=0)}(x) &= 2\sqrt{\frac{2}{a}}\left[-2\cos(y)+4\cos^3(y)\right]\sin(y) && (44)\\
\psi_4^{(S=0)}(x) &= \sqrt{\frac{2}{a}}\left[1-12\cos^2(y)+16\cos^4(y)\right]\sin(y)\,, && (45)
\end{aligned}$$

which are indeed of the same general form.

Building on the similarity between these results and the HO case, we argue that the $\sin^{(S+1)}(y)$ terms here play a role akin to the $e^{-x^2/2\beta^2}$ factors in the HO case, being responsible for 'enforcing the boundary conditions.' In the SISW case, the $\sin^{(S+1)}(y)$ components enforce the boundary conditions at the $x = 0, a$ infinite walls, while for the oscillator solutions the Gaussian factors guarantee the smooth vanishing of the wave function at $x = \pm\infty$.

Motivated by this similarity, we attempt to factor out the $\sin^{(S+1)}(y)$ dependence by writing

$$\psi_n^{(S)}(x) = G_n^{(S)}(y)\,\sin^{S+1}(y)\,, \tag{46}$$

and substituting it into the (dimensionless) Schrödinger equation for $V^{(S)}(x)$, namely

$$\frac{d^2\psi_n^{(S)}(y)}{dy^2} - \frac{S(S+1)}{\sin^2(y)}\psi_n^{(S)}(y) + (n+S+1)^2\psi_n^{(S)}(y) = 0\,, \tag{47}$$

thereby obtaining a differential equation for the $G_n(y)$ components given by

$$\sin(y)\frac{d^2G_n^{(S)}(y)}{dy^2} + 2(S+1)\cos(y)\frac{dG_n^{(S)}(y)}{dy} + \left[(n+S+1)^2-(S+1)^2\right]\sin(y)G_n^{(S)}(y) = 0\,. \tag{48}$$

Using our experience with the form of the solutions for general (n, S), from Eqs. (36) - (40) and (41) - (45) and beyond, we assume that $G_n^{(S)}(y)$ can be expanded in a (presumably finite) series in powers of $\cos(y)$, by writing

$$G_n^{(S)}[\cos(y)] = \sum_{k=0}^{\infty} a_{k,n}\cos^k(y)\,. \tag{49}$$

Substituting this into Eq. (48) we find

$$\sum_k k(k-1)a_{k,n}\cos^{k-2}(y) = \sum_k a_{k,n}[(k+S+1)^2 - (n+S+1)^2]\cos^k(y)\,, \tag{50}$$

and upon relabeling and comparing similar powers of $\cos(y)$ we find the recursion relation amongst the expansion coefficients

$$a_{k+2,n} = a_{k,n}\left[\frac{(k+S+1)^2 - (n+S+1)^2}{(k+1)(k+2)}\right]\,. \tag{51}$$

This expression confirms that for a given n, the series in $\cos(y)$ will indeed terminate with a highest power of $k = n$. It also connects every **other** term in the expansion, implying that starting with arbitrary $a_{0,n}, a_{1,n}$, separate even and odd series will be generated.

This is the identical logic used to conclude that the series expansion for the harmonic oscillator (HO) problem must reduce to a finite polynomial, since otherwise the infinite series would yield the incorrect behavior as $x \to \pm\infty$. In HO case, the recursion relation of the coefficients one obtains also connects every other term in the expansion, also giving the expected even and odd parity solutions.

The generality of the results obtained so far, namely that the solutions can be constructed from simple factors which encode the behavior of the solutions at the boundaries, along with polynomials (here in the variable $\cos(y)$) which describe the dynamical behavior inside the well (the 'wiggliness' if you will) suggested to us that these expressions might be able to be mapped onto existing forms in the mathematical literature. Given that the equation for the polynomials yields solutions involving the variable $w = \cos(y)$, we rewrite the differential equation for $G_n^{(S)}(y) = F_n(w)$ and obtain

$$\sin^2(y)F_n''(w) - \cos(y)F_n'(w)(2S+3) + n(n+2S+2)F_n(w) \;=\; 0 \tag{52}$$

or

$$(1-w^2)F_n''(w) - wF_n'(w)(2S+3) + n(n+2S+2)F_n(w) \;=\; 0\,. \tag{53}$$

This final form is indeed known in the mathematical physics literature[20] as being the equation for the Gegenbauer polynomials, sometimes written in the form

$$(1-z^2)F''(z) - zF'(z)(2\alpha+1) + n(n+2\alpha)F(z) = 0\,, \tag{54}$$

with solutions expressed in the notation $F(z) = C_n^\alpha(z)$, where we associate $\alpha = S+1$.

The Gegenbauer functions are polynomials of order n, defined over the interval $z \in (-1,+1)$, and mutually orthogonal under the weight $(1-z^2)^{\alpha-2}$. They have n nodes over the allowed range and as x varies from 0 to a in our physical problem, the argument of $C_n^{\alpha}[\cos(\pi x/a)]$ varies in the defined range of $(-1,+1)$. The appearance of such orthogonal polynomials should be very familiar from the 1D harmonic oscillator, where the Hermite polynomials, $H_n(z)$, appear with the weight being e^{-z^2} defined over the interval $(-\infty,+\infty)$, and very similar results from the 3D Coulomb problem.

Integrals over the products of the $C_n^{\alpha}(z)$ times the appropriate weight functions[20] are exactly the type of results needed to determine the normalization of the solutions. Specifically, we find using such results that we can write the general (n,S) solution for the SISW hierarchy of potentials in the form

$$\psi_n^{(S)}(x) = \frac{1}{\sqrt{a}}\left[\frac{2^{2S+1}\Gamma(n+1)\Gamma^2(S+1)(n+S+1)}{\Gamma(n+2S+2)}\right]^{1/2} C_n^{(S+1)}[\cos(y)]\sin^{S+1}(y)\,, \tag{55}$$

where again $y \equiv \pi x/a$. Using the standard results for the lowest lying Gegenbauer polynomials, namely $C_0^{\alpha}(y)=1$ and $C_1^{\alpha}(y)=2\alpha y$, we can also reproduce our earlier 'experimentally derived' results for $n=0,1$ for all S, including the normalization factors, in Eqs. (31) and (32). In addition, for $S=0$, the results of Eq. (55) reduce to a new form of the ISW wave functions

$$\psi_n^{(S=0)}(x) = \psi_n^{\mathrm{ISW}}(x) = \sqrt{\frac{2}{a}}\,\mathcal{U}_{n-1}[\cos(y)]\;\sin(y)\,, \tag{56}$$

where $\mathcal{U}_n$ are the Chebyshev polynomials of the second kind. We also note here that the approach leading up to Eq. (51) can also be used for this new version for the ISW. We can also solve the time-independent Schrödinger equation by asserting that solutions of the differential equation are in the form polynomial in $\cos(y)$:

$$\psi_n(x) = G_n[\cos(y)]\sin(y)\,, \tag{57}$$

where

$$G_n^{(S=0)}[\cos(y)] = \sum_{k=0}^{n} a_{k,n}\cos^k(y)\,. \tag{58}$$

Upon inserting this trial wave function into the time-independent Schrödinger equation, we obtain a differential equation for $G_n^{(S)}(y)$ given by

$$\sin(y)\frac{d^2G_n^{(S)}(y)}{dy^2} + 2\cos(y)\frac{dG_n^{(S)}(y)}{dy} + \left[(n+1)^2-1\right]\sin(y)G_n^{(S)}(y) = 0\,. \tag{59}$$

Similar to Eq. (48), we take $S = 0$ to represent the ISW. We then substitute Eq. (58) into Eq. (59) to find that

$$\sum_k k(k-1)a_{k,n}\cos^{k-2}(y) = \sum_k a_{k,n}[(k+1)^2 - (n)^2]\cos^k(y) \tag{60}$$

Simplifying this expression allows us to obtain a recursive relationship for the constant coefficient in Eq. (60)

$$a_{k+2,n} = \frac{(k+1)^2 - n^2}{(k+2)(k+1)} a_{k,n} \ , \tag{61}$$

where separate even and odd series will be generated. If $a_{0,1} = 1$ and $a_{0,2} = 2$, Eq. (61) generates the Chebyshev polynomials of the second kind in Eq. (58), as expected. We also note that when $S = 0$, Eq. (51) reduces to Eq. (61).

The expression in Eq. (55) is one we have not found in the physics or mathematics literature and which we believe to be a new result. Given the simplicity of its form, we suggest that this system may find a useful place in the mathematical physics literature of quantum mechanics, especially given the array of additional questions (see Sec. V for examples) one can then pursue in analyzing its structure.

While we were introduced to the problem leading to this final result using the methods of SUSYQM outlined in Sec. II, we found the standard methods of analysis used in many quantum mechanics textbooks (extracting the behavior at the boundary conditions, series expansion of the 'remainder' problem, etc.) absolutely necessary to make the connection to well-known special functions to arrive at the final compact and closed-form general solution for all (n, S) in Eq. (55) which is the main result of this work. Instructors and students alike can certainly appreciate seeing decades-old textbook methodologies applied to a new class of quantum mechanical problems in a very pedagogical manner. We wish to emphasize again the utility of applying powerful symbolic manipulation techniques to new theory problems, somewhat in the same way that experimentalists would apply statistical/visualization techniques to data to look for patterns.

V. CONCLUSIONS AND FUTURE DIRECTIONS

In this work, we have focused on obtaining the solutions to a novel set of quantum mechanical problems encoded in the hierarchy of supersymmetric extensions of the most familiar of all textbook models, the infinite square well. Using the methods of supersymmetric quantum mechanics, and the mathematical tools outlined in almost every undergraduate quantum textbook, we have been able to present elegant, compact, closed-form solutions to a new class of quantum-mechanical potentials, dramatically extending earlier discussions[4,9,10] of this system. While deriving these (seemingly novel) results, we have emphasized the interplay between various solutions methods in quantum mechanics when approaching new problems, in the same way that any student might when facing 'familiar' problems for the first time, so in that sense our work is very pedagogical.

This model system is now ripe for further study, with many additional areas of research to explore or pedagogical application to use in the classroom. For example, with the ability to now easily calculate many physical quantities of interest, we have been able to find closed form expressions for the expectation values of the potential and kinetic energies in a general (n, S) state, namely

$$\langle \psi_n^{(S)} | V^{(S)}(x) | \psi_n^{(S)} \rangle = \mathcal{E}_0 \left\{ \frac{2S(S+1)(n+S+1)}{(2S+1)} \right\} \tag{62}$$

$$\frac{1}{2m} \langle \psi_n^{(S)} | \hat{p}^2 | \psi_n^{(S)} \rangle = \langle \psi_n^{(S)} | \hat{T} | \psi_n^{(S)} \rangle = \mathcal{E}_0 \left\{ \frac{[(2S+1)n + (S+1)](n+S+1)}{(2S+1)} \right\} , \tag{63}$$

where we find that

$$\langle \psi_n^{(S)} | V^{(S)}(x) | \psi_n^{(S)} \rangle + \langle \psi_n^{(S)} | \hat{T} | \psi_n^{(S)} \rangle = \mathcal{E}_0 (n+S+1)^2 = E_n^{(S)} . \tag{64}$$

We have also confirmed (by explicit calculation) that the virial theorem holds, namely that

$$\langle \psi_n^{(S)} | \hat{T} | \psi_n^{(S)} \rangle = \frac{1}{2} \langle \psi_n^{(S)} \left| x \frac{dV^{(S)}(x)}{dx} \right| \psi_n^{(S)} \rangle , \tag{65}$$

for $S \geq 1$ where the potential energy function, $V^{(S)}(x)$, is better behaved than $V^{(S=0)}(x) = V^{\text{ISW}}(x)$. Problems such as these (and many more which suggest themselves) can be used as new classroom examples or homework assignments (not appearing in standard textbooks) and therefore can certainly be incorporated into an advanced undergraduate class, especially one where computer math tools are encouraged as an educational tool.

The simple form of the $\psi_n^{(S)}(x)$ in terms of known special functions suggests that the momentum-space wave functions might also be written in equally compact and elegant ways. For example, for the 3D Coulomb problem (hydrogen atom) the momentum-space solutions were deftly derived in the very early days of quantum mechanics[21] in terms of known special functions, in fact Gegenbauer polynomials! We have already started to explore the general $\phi_n^{(S)}(p)$ solutions and have confirmed that they exhibit large $|p|$ behavior given by $|\phi_n^{(S)}(p)| \sim p^{-(2+S)}$, consistent with theorems[22] connecting the discontinuities of $\psi(x)$ (here encoded in the increasingly smooth x^{S+1} behavior of the wave functions at the walls) very directly to the large momentum limit of $\phi(p)$. There are likely many closed-form results waiting to be uncovered in the continued mathematical physics analysis of both the position-space and momentum-space versions of this problem.

One of the most striking results of the SISW hierarchy is the simple form of the general $V^{(S)}(x)$ potentials, and especially their explicit dependence on the $S(S+1)$ factor. While we expect the general form in Eq. (33) near any infinite wall in a SUSY hierarchy, the fact that the $S(S+1)$ factor appears as a pre-factor to a relatively simple functional form is perhaps surprising. We note that two earlier works have already explored the behavior of 'half-potential' problems, ones defined by

$$\tilde{V}(x) \equiv \begin{cases} \infty & \text{for } x < 0 \\ V(x) & \text{for } 0 < x \end{cases} , \tag{66}$$

for cases where $V(x)$ has a very high degree of symmetry in the complete 1D case, namely for the 'half-oscillator'[23] and the 1D Coulomb problem.[24] The authors of those studies have considered (in passing) the $S = 1$ supersymmetric extensions of the $S = 0$ original potentials for each case and have found

$$V^{(S=0)}(x) = \frac{1}{2}m\omega^2 x^2 \quad \Longrightarrow \quad V^{(S=1)}(x) = \frac{1}{2}m\omega^2 x^2 + \frac{2\hbar^2}{2mx^2} , \tag{67}$$

$$V^{(S=0)}(x) = -\frac{Ke^2}{x} \quad \Longrightarrow \quad V^{(S=1)}(x) = -\frac{Ke^2}{x} + \frac{2\hbar^2}{2mx^2} , \tag{68}$$

where the $S = 1$ results have not just an approximate $S(S+1)\hbar^2/2mx^2$ behavior near the infinite wall boundary (as suggested by Eq. (33)), but an exact 'centrifugal' term for all $x > 0$. We have extended those results and find that repeated symmetrizations of these two systems give

$$V^{(S=0)}(x) = \frac{1}{2}m\omega^2 x^2 \quad \Longrightarrow \quad V^{(S)}(x) = \frac{1}{2}m\omega^2 x^2 + \frac{S(S+1)\hbar^2}{2mx^2}\,, \tag{69}$$

$$V^{(S=0)}(x) = -\frac{Ke^2}{x} \quad \Longrightarrow \quad V^{(S)}(x) = -\frac{Ke^2}{x} + \frac{S(S+1)\hbar^2}{2mx^2} \tag{70}$$

for the general S case. We have also found closed-form expressions for the general solutions of these systems, using results from the related fully three-dimensional versions of the harmonic oscillator and Coulomb problem, where similar potentials occur in the corresponding radial equation. In these cases, the role of the S parameter is indeed closely related to the angular momentum quantum number (l) of the 3D problem.

VI. ACKNOWLEDGEMENTS

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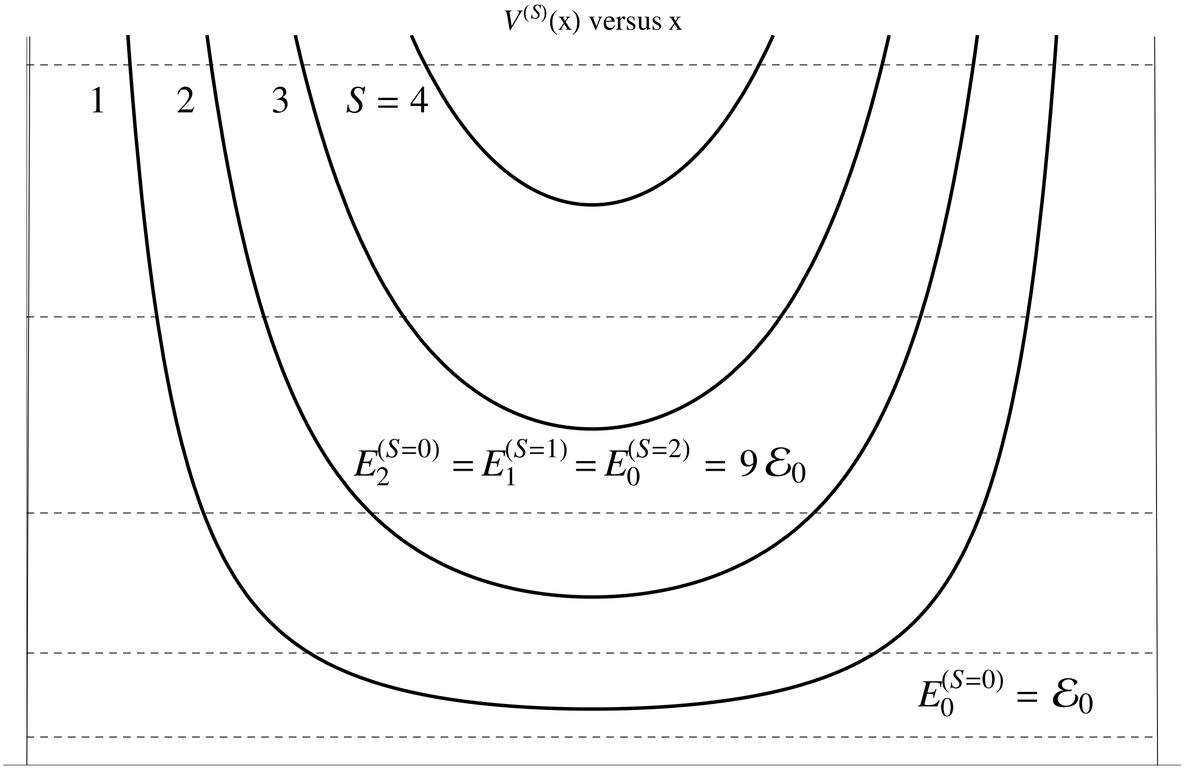

FIG. 1: Superpartner potentials, $V^{(S)}(x)$ versus x, for $S = 0$ (infinite square well or ISW with infinite walls at $x = 0, a$) and $S = 1, 2, 3, 4$ (solid curves), along with low-lying energy levels. The ground-state energy of the ISW, $E_0^{(S=0)} = \mathcal{E}_0$, is shown as the bottom horizontal dashed line and the ISW is the only state for which that is a solution. Higher energy levels, such as the one labeled $E_2^{(S=0)} = E_1^{(S=1)} = E_0^{(S=2)} = 9\mathcal{E}_0$, appear as solutions for more than one value of S.

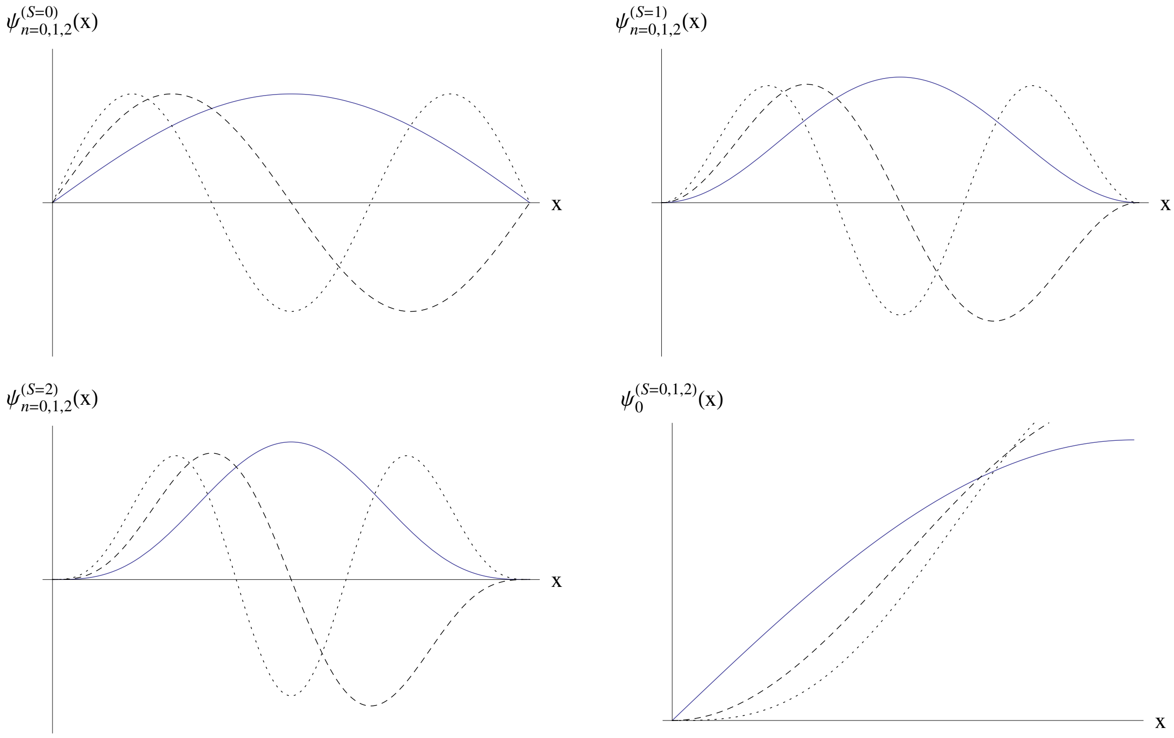

FIG. 2: Position-space solutions, $\psi_n^{(S)}(x)$ versus x, for $n = 0, 1, 2$ (solid, dashed, dotted curves) for $S = 0$ or ISW case (upper left), $S = 1$ (upper right), $S = 2$ (lower left). In the lower right we show the ground-state solutions, $\psi_0^{(S)}(x)$, for $S = 0, 1, 2$ (solid, dashed, dotted curves) to illustrate the x^{S+1} behavior near the infinite walls, as described near the end of Sec. III.

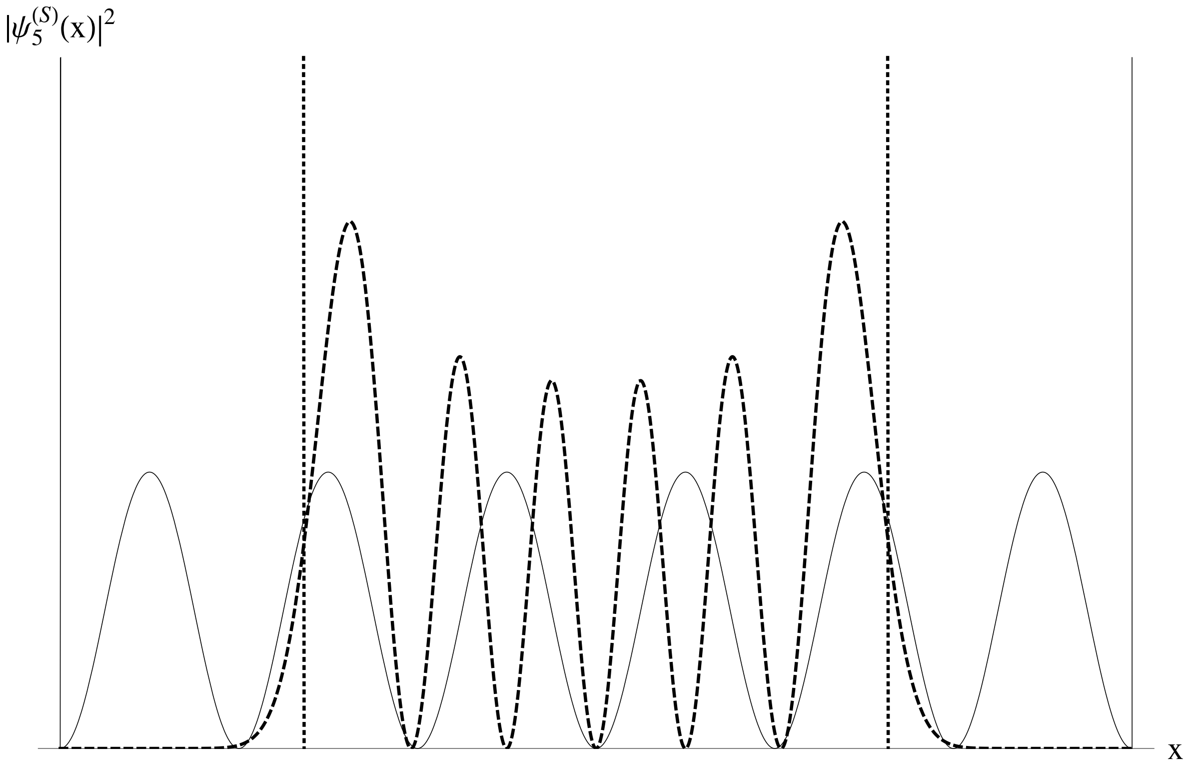

FIG. 3: Probability density, $|\psi_n^{(S)}(x)|^2$ versus x, for $n = 5$, for $S = 0, 10$ (solid, dashed curves). For the $S = 0$ or ISW case, the probability density approaches the classical 'flat' limit (after local averaging) for large n, while for $S > 0$, the peaking of the probability density near the classical turning points of $V^{(S)}(x)$ is clear. For example, the bold vertical dotted lines indicate the classical turning points for the $(n, S) = (5, 10)$ case.